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On bound states of photons in noncommutative U(1) gauge theory

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Abstract. We consider the possibility that photons of noncommutative U(1) gauge theory can make bound states. Using the potential model, developed based on the constituent gluon picture of QCD glue-balls, arguments are presented in favor of the existence of these bound states. The basic ingredient of the potential model is that the self-interacting massless gauge particles may get mass by the inclusion of non-perturbative effects.

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1 Introduction

In Abelian gauge theories on ordinary space-time, there is no self-coupling between the gauge fields. The best known example is the quantum theory of the interaction between electric charges and photons. The situation is different in non-Abelian theories, and due to the commutator term in the field strength, these theories entail a direct interaction between the quanta of gauge fields. It is now widely believed that the strong interaction is described by a non-Abelian gauge theory accompanied with proper matter fields, quarks, the so-called QCD. As gluons are the quanta of QCD gauge field, from the very beginning the possibility was considered that gluons can make bound states free from valance quarks, the so-called glue-balls. Although the properties of glue-balls have been studied for a long time, their existence have not still been proved experimentally.

Recently great interest has grown in the study of field theories on spaces whose coordinates do not commute. These spaces, as well as the field theories defined on them, are known by the names of non-commutative spaces and theories. In contrast to U(1) gauge theory on ordinary space-time, as we briefly review in next section, the non-commutative version of the theory involves direct interactions between photons. Interestingly one finds the situation to be very reminiscent of that of non-Abelian gauge theories, and then the question is whether there are some kinds of bound states in analogy with glue-balls of QCD, which here might be called "photo-balls". It is this question that we consider in this work. Our approach to study photo-balls is based on one of the methods that has been developed for glue-balls. As glue-balls are non-perturbative in nature, there is still no systematic way for the calculation of their properties from the first principles of QCD. Instead, over the years many approaches have been developed for extracting the glue-ball's properties, though each approach is based on expectations or estimate calculations.

Among many others, one approach for studying the properties of glue-balls has been the so-called constituent gluon model. In any study of bound states of gluons, one encounters a situation in which gluons, though at first they were introduced as massless to the Lagrangian, are bound and do not disjoint to propagate to infinity. Correspondingly, it is argued that quantum fluctuations around a charged particle, that should be treated non-perturbatively in QCD, can make an accompanied cloud for it, causing a dynamically generated mass [1,2]. Accordingly, it appears to be very useful to define constituent quarks and gluons, for which we assume a mass of the order of bound states of the theory, while their Lagrangian counterparts may be massless or almost massless. As extracting the masses of constituent particles from first principles has not yet been done in a satisfactory way, the best evaluations come from estimations based on general considerations, phenomenology or lattice calculations.

Once one accepts that a glue-ball is a bound state of constituent gluons, the question is what the effective theory is that captures the interaction between them. One approach is to consider constituent gluons as massive quanta of an effective gauge theory. It needs some kinds of proof, but hopefully this effective gauge theory has the same qualitative features as the true (massless) theory, but in the non-perturbative regime [1,2]. Since it is

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believed that the main contribution to the mass of a glue-ball is coming from the constituent masses of gluons, it is expected that constituent gluons move non-relativistically inside the glue-ball, and so perturbative calculations for finding the effective potential should be done in the non-relativistic regime. Having the effective potential at hand, by studying the Schrödinger-type equations, one can make estimations about the mass or size of glue-balls. It is the heart of the potential model approach for studying the properties of the glue-balls [3–5].

There are two related issues when we are considering the effective gauge theory of constituent gluons as massive vector particles. First, it is known that the gauge symmetry is lost via the mass term, and the second is that massive gauge theories are known to be perturbatively non-renormalizable. Here we give comments on these issues [1, 2]. The non-renormalizability of massive gauge theories is under this assumption that the mass in the theory appears as a fixed parameter, surviving at large momentum. In fact the insufficient decrease of the propagator of a massive vector particle at large momentum, due to simple power counting, suggests that the theory cannot be renormalizable. But the situation might be different in a theory with constituent mass. At very large momentum, where the coupling constant is small due to asymptotic freedom, the perturbation is valid and gluons appear as massless particles. So the mass of a constituent gluon, which is generated dynamically, depends on the momentum and vanishes at large momentum. In a theory for gluons, it is argued that if one can keep the dependence of constituent mass on momentum, which of course is possible only by including the non-perturbative effects, the theory may appear to be non-perturbatively renormalizable.

Although the argument above is for a model involving dynamically generated mass, due to lack of a systematic treatment of non-perturbative effects, much can be learned via a kinematical description of the gluon mass [2]; this is to assume the mass as a fixed parameter, though a problem still remains with local gauge symmetry. To overcome this problem, there is a prescription that we review briefly below. The starting point for the QCD case is the Lagrangian density [1,2]

$$\begin{aligned} L = & -\frac{1}{4}\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) \\ & + \frac{1}{2}m^2\mathrm{Tr}\left(A_\mu - \frac{1}{g}V(\varphi)\partial_\mu V^\dagger(\varphi)\right)^2, \end{aligned} \tag{1}$$

in which

$$V(\varphi) = \exp\left[\frac{\mathrm{i}}{2}\sum_a T^a\varphi_a\right] \tag{2}$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]$, with $[T^a, T^b] = \mathrm{i}f_c^{ab}T^c$. The action is invariant under

$$A'_\mu = UA_\mu U^{-1} + \frac{1}{g}U\partial_\mu U^{-1}, \quad V(\varphi') = UV(\varphi), \tag{3}$$

in which $U = U(x)$ is the unitary matrix defining the transformation. Now we see that, although the gauge fields have got mass, the local symmetry is kept. Of course we mention that giving mass is done paying the price of introducing extra scalar fields. We have another example of this observation in the spontaneous symmetry breaking mechanism, in which we are left with Goldstone bosons. In fact, these extra scalar fields, *just* like their Goldstone boson counterparts, do not appear in the S-matrix, i.e. as external legs of diagrams. One can insert the scalar fields into the Lagrangian via a power series solution in g [1,2]:

$$\varphi_a = g\frac{1}{\Box}\partial\cdot A_a - g^2[\cdots]_a, \tag{4}$$

getting a non-local but still gauge invariant theory involving only A_μ's. This mechanism has been used for the case of a constituent gluon description of QCD glue-balls [3–5], and here we use it for photo-balls of non-commutative U(1) theory.

As mentioned above the extra scalars do not appear as external legs of diagrams, but the situation is even simpler as far as one considers just the tree diagrams, in which one can ignore the scalars. So for tree diagrams, and in a proper gauge, the Lagrangian density in use is practically [2,4,5]

$$L = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}m^2A^{a\mu}A^a_\mu, \tag{5}$$

simply as a gauge theory for massive gluons.

In the non-relativistic limit the potential can be read off from the total invariant amplitude $\mathcal{M}_{fi}$ via the Fourier transform [3–5]

$$V(\mathbf{r}) = \int\frac{\mathrm{d}^3q}{8\pi^3}\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}}{4\sqrt{E_1E_2E_3E_4}}\mathrm{i}\mathcal{M}_{fi}, \tag{6}$$

in which $\mathbf{q}$ is the momentum transferred between the incoming particles. The total invariant amplitude gets a contribution at tree level from the s-, t- and u-channels, and the so-called seagull (s.g.) diagram, coming from the four-gluon vertex of QCD [3–5]. In the non-relativistic limit it can be shown that the s-channel's contribution is negligible, so one gets the final expression [3–5]

$$\begin{aligned} \mathrm{i}\mathcal{M}_{fi} = & \frac{\mathrm{i}g^2f^{ace}f^{bde}}{\mathbf{q}^2+m^2} \\ & \times\left(4m^2 + 3\mathbf{q}^2 - 2\mathbf{S}^2\mathbf{q}^2 + 2(\mathbf{S}\cdot\mathbf{q})^2 + 6\mathrm{i}\mathbf{S}\cdot(\mathbf{q}\times\mathbf{p}_i)\right) \\ & - \mathrm{i}g^2\left(f^{abe}f^{cde} - f^{ace}f^{bde}\left(\frac{1}{2}\mathbf{S}^2 - 2\right)\right). \end{aligned} \tag{7}$$

The organization of the rest of this work is as follows. In Sect. 2 we review some basic features of canonical non-commutative spaces, and also the field theories defined on them, specially non-commutative U(1) gauge theory. We also make remarks on some aspects of non-commutative U(1) gauge theory that make this theory to some extent similar to QCD. Section 3 mainly contains the dynamics of photons under the effective potential obtained in Appendix A. The existence proof of bound states is also presented in Sect. 3. Section 4 is for our conclusion and discussion. Appendix A is devoted to extracting the effective potential between massive photons, by studying the non-relativistic behavior of their scattering. The distributional derivatives are presented in Appendix B.

2 Non-commutative space-time and U(1) theory

Over the last years a great deal of attention has been devoted to the formulation and the study of field theories on non-commutative spaces. One of the original motivations has been to get finite field theories via the intrinsic regularizations which are encoded in some of the non-commutative spaces [6]. The other motivation goes back to the natural appearance of non-commutative spaces in some areas of physics, and the recent one in string theory. It has been understood that string theory involves some kinds of non-commutativities; two examples are
(1) the coordinates of bound states of N D-branes are presented by $N \times N$ Hermitian matrices [7], and
(2) the longitudinal directions of D-branes in the presence of a B-field background appear to be non-commutative, as is seen by the ends of open strings [8–10]. In the latter case for a constant background one simply gets the canonical non-commutative space-time, introduced by the commutation relations for coordinates as follows:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}. \tag{8}$$

Since the coordinates do not commute, any definition of functions or fields should be performed under a prescription for ordering of the coordinates, and one choice can be the symmetric one, the so-called Weyl ordering. To any function $f(x)$ on ordinary space, one can assign an operator $\hat{O}_f$ by

$$\hat{O}_f\,(\hat{x}) := \frac{1}{(2\pi)^n}\int \mathrm{d}^n k \tilde{f}(k)\mathrm{e}^{-\mathrm{i}k\cdot\hat{x}}, \tag{9}$$

in which $\tilde{f}(k)$ is the Fourier transform of $f(x)$ defined by

$$\tilde{f}(k) = \int \mathrm{d}^n x f(x)\mathrm{e}^{\mathrm{i}k\cdot x}. \tag{10}$$

Due to the presence of the phase $\mathrm{e}^{-\mathrm{i}k\cdot\hat{x}}$ in the definition of $\hat{O}_f$, we recover the Weyl prescription for the coordinates. In a reverse way we also can assign to any symmetrized operator a function or field living on the non-commutative plane. Also, we can assign to the product of any two operators $\hat{O}_f$ and $\hat{O}_g$ another operator by

$$\hat{O}_f \cdot \hat{O}_g =: \hat{O}_{f\star g}, \tag{11}$$

in which f and g are multiplied under the so-called $\star$-product defined by

$$(f\star g)(x) = \exp\left(\frac{\mathrm{i}\theta^{\mu\nu}}{2}\partial_{x_\mu}\partial_{y_\nu}\right) f(x)g(y)\mid_{y=x}. \tag{12}$$

By all this one learns how to define physical theories on non-commutative space-time, and eventually it appears that the non-commutative field theories are defined by actions that are essentially the same as in ordinary space-time, with the exception that the products between fields are replaced by the $\star$-product; see [11] for a review. Though the $\star$-product itself is not commutative (i.e., $f\star g \neq g\star f$) the following identities make some of the calculations easier:

$$\int f\star g \mathrm{d}^n x = \int g\star f \mathrm{d}^n x = \int fg \mathrm{d}^n x,$$
$$\int f\star g\star h \mathrm{d}^n x = \int f(g\star h)\mathrm{d}^n x = \int (f\star g)h \mathrm{d}^n x,$$
$$\int f\star g\star h \mathrm{d}^n x = \int h\star f\star g \mathrm{d}^n x = \int g\star h\star f \mathrm{d}^n x.$$

By the former two ones we see that in integrands always one of the stars can be removed. Besides it can be seen that the $\star$-product is associative, i.e., $f\star g\star h = (f\star g)\star h = f\star(g\star h)$, and so it is not important which two ones should be multiplied first.

The pure gauge field sector of non-commutative U(1) theory is defined by the action

$$S_{\text{gauge-field}} = -\frac{1}{4}\int \mathrm{d}^4 x F_{\mu\nu}\star F^{\mu\nu} = -\frac{1}{4}\int \mathrm{d}^4 x F_{\mu\nu}F^{\mu\nu}, \tag{13}$$

in which the field strength $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \mathrm{i}e[A_\mu(x), A_\nu(x)]_\star; \tag{14}$$

by definition $[f,g]_\star = f\star g - g\star f$. We mention $[x^\mu, x^\nu]_\star = \mathrm{i}\theta^{\mu\nu}$. The action above is invariant under local gauge symmetry transformations:

$$A'_\mu(x) = U\star A_\mu(x)\star U^{-1} + \frac{\mathrm{i}}{e}U\star\partial_\mu U^{-1}, \tag{15}$$

in which $U = U(x)$ is the $\star$-phase, defined by the function $\lambda(x)$ via the $\star$-exponential:

$$U(x) = \exp_\star(\mathrm{i}\lambda) = 1 + \mathrm{i}\lambda - \frac{1}{2}\lambda\star\lambda + \cdots, \tag{16}$$
$$U\star U^{-1} = U^{-1}\star U = 1, \tag{17}$$

in which $U^{-1} = \exp_\star(-\mathrm{i}\lambda)$. Under the above transformation, the field strength transforms as

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = U\star F_{\mu\nu}\star U^{-1}. \tag{18}$$

We mention that the transformations of the gauge field as well as the field strength look like those of non-Abelian gauge theories. Besides we see that the action contains terms which are responsible for the interaction between the gauge particles, again as the situation we have in non-Abelian gauge theories. We see how the non-commutativity of coordinates induces properties of the fields and their transformations, as if they belonged to a non-Abelian theory; the subject of how the characters of coordinates and fields may be related to each other is discussed in [12]. These observations make it reasonable to study whether and how the photons can make up bound states in such a theory.

There is another observation that promotes the formal similarities of non-commutative U(1) and non-Abelian theories as to their behaviors; that is, the negative sign of the β-function, which makes manifest that non-commutative pure U(1) gauge theory is asymptotically free [13,14]. By this it is more reasonable to see if the techniques developed for QCD purposes can also be used for non-commutative U(1) theory.

The phenomenological implications of possible non-commutative coordinates have been the subject of a very large number of research publications in the last years. Among many others, here we can give just a brief list of works, specially those concerning the phenomenological implications of non-commutative U(1) theory. The effect of the non-commutativity of space-time is studied for possible modifications that may appear in the high energy scattering amplitudes of particles [15], in the energy levels of light atoms [16,17], and the anomalous magnetic moment of the electron [18]. The ultra-high energy scattering of massless photons of non-commutative U(1) theory is considered in [19] and the tiny change in the total amplitude is obtained as a function of the total energy. Some other interesting features of non-commutative ED and QED are discussed in [20]. The issue of the formation of new bound states by space-time non-commutativity has been considered in [21].

3 On the existence of bound states

Having the effective potential, the starting point for studying the bound state problem is the Schrödinger-type equation by the Hamiltonian:

$$H = 2m + H_{2\mathrm{b}}, \tag{19}$$

in which m is the constituent mass, and $H_{2\mathrm{b}}$ is for the Hamiltonian capturing the dynamics of a two-body system. For example, in the glue-ball case $H_{2\mathrm{b}}$ usually consists three parts: the kinetic term, the potential term coming from perturbative calculations and the string potential. The string potential usually is taken in the form $V_{\mathrm{string}} = 2m(1 - \mathrm{e}^{-\beta r})$, in which β is related to the tension of string stretched between the gluons. The formation of strings is expected from simulations on lattice, as well as the confinement hypothesis [3–5]. Due to lack of analytical solutions, approximation methods, specially the variational method, appear to be practically useful [3–5]. We mention that without any reliable estimation of the value of the constituent mass, all efforts for the evaluation of bound state properties, such as the mass and size, do not get any definitive result. There have been lots of theoretical and numerical efforts, like those done using the lattice version of the theory, together with phenomenological expectations, to estimate the mass of the constituent gluons.

In Appendix A an effective potential between two photons is obtained based on the constituent picture of glue-balls described in the Introduction. According to this picture, as we expect that in bound states gluons appear to be massive due to non-perturbative effects, the effective theory for studying the dynamics of gluons practically is a massive gauge theory, as far as one is concerned with tree diagrams. Then by studying the non-relativistic limit of the scattering amplitude between two massive gluons one can extract an effective potential between scattered particles. This is the same approach as we use for extracting an effective potential between photons of the non-commutative theory.

Comparing to the case with glue-balls, the situation is more difficult in any study of photo-balls of non-commutative U(1) theory. First, by the present experimental data we just can suggest an upper limit for non-commutative effects, leaving θ unspecified. Second, at present neither can we say anything about the value of constituent mass, nor about how it varies with other parameters, specially θ. In this sense, no study can yield a definitive result or suggestion for the quantities we like to know of photo-balls.

Here we try to formulate the dynamics based on the effective potential obtained in Appendix A. Based on this formulation, we specially present a proof of the existence of the bound states. Since the issue of the possible formation of string-like objects in non-commutative U(1) theory is not in a conclusive situation, we do not consider a string potential in this work. We recall that by including the string potential the existence proof of bound states would be a trivial task. Also as the potential (A.20) is very complicated, for the study of the possible bound states, we restrict ourselves here to the zero total-spin case, the $S = 0$ case; we also ignore the terms coming from the distributional derivatives, the so-called D.D. terms. So we have the potential (A.21), that is,

$$V_{2\gamma}^{S=0}(\mathbf{r}) = \frac{e^2}{4}\frac{\mathrm{e}^{-mr}}{4\pi}\left[-\lambda^2\frac{mr+1}{r^3} + (\boldsymbol{\lambda}\cdot\hat{\mathbf{r}})^2\frac{m^2r^2+3mr+3}{r^3}\right], \tag{20}$$

in which we have the vector $\boldsymbol{\lambda} = \frac{1}{2}\mathbf{p}\times\boldsymbol{\theta}$, with $\mathbf{p}$ as the momentum of one of the photons in the center-of-mass frame, and $\boldsymbol{\theta}$ as a vector built up from the non-commutative tensor θ^{ij}; see Appendix A. m is the supposed constituent mass of the photons. The interpretation of potential above as the potential between two electric dipoles in a theory with massive exchange particles is presented in Appendix A. For the sake of definiteness, we take the vector $\boldsymbol{\theta}$ in the z direction, that is $\boldsymbol{\theta} = \theta\mathbf{z}$. It is more convenient to work in cylindrical coordinates (ρ, ϕ, z), in which the kinetic energy, recalling that the effective mass in relative motion is $m/2$, is $T = \frac{1}{2}\frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$. Then we have

$$\lambda^2 = \frac{1}{4}\left(\theta^2p^2 - (\boldsymbol{\theta}\cdot\mathbf{p})^2\right) = \frac{1}{4}\theta^2(p_x^2 + p_y^2) = \frac{1}{16}m^2\theta^2(\dot{\rho}^2 + \rho^2\dot{\phi}^2), \tag{21}$$

and also

$$(\boldsymbol{\lambda}\cdot\hat{\mathbf{r}})^2 = \frac{1}{r^2}(\boldsymbol{\lambda}\cdot\mathbf{r})^2 = \frac{1}{16r^2}m^2\theta^2\rho^4\dot{\phi}^2, \tag{22}$$

in which r is the distance between two photons, $r = \sqrt{\rho^2 + z^2}$. We see that, while the contribution coming from the velocity $\dot{\rho}$ always yields an attractive force, the contribution from the angular velocity $\dot{\phi}$ depends on the ratio ρ^2/r^2; it could be attractive or repulsive. In fact the ratio ρ^2/r^2, as representing how much the photons move off from the plane $z = 0$, also determines the relative orientation between $\mathbf{r}$ and the components of the electric dipoles generated due to the velocity $\dot{\phi}$. We recall that the relative orientation of dipoles and the position vector appears in the dipole–dipole potential (A.22). By all this we have the Lagrangian

$$\begin{aligned} L &= T - V \\ &= \frac{1}{4} m \left[1 + a f_1(r)\right] \dot{\rho}^2 + \frac{1}{4} m \dot{z}^2 \\ &+ \frac{1}{4} m \rho^2 \left[1 + a \left(f_1(r) - \rho^2 f_2(r)\right)\right] \dot{\phi}^2, \end{aligned} \tag{23}$$

in which $a = \frac{e^2}{64\pi} m \theta^2$ is a constant, and

$$\begin{aligned} f_1(r) &= \mathrm{e}^{-mr} \frac{mr + 1}{r^3}, \\ f_2(r) &= -\frac{1}{r} \frac{\partial f_1}{\partial r} = \mathrm{e}^{-mr} \frac{m^2 r^2 + 3mr + 3}{r^5}. \end{aligned} \tag{24}$$

We mention that the first two terms are positive definite, while the third one can be negative, zero and positive. The coordinate ϕ is cyclic, and hence its momentum, given by

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2} m \rho^2 \left[1 + a \left(f_1(r) - \rho^2 f_2(r)\right)\right] \dot{\phi} = K, \tag{25}$$

is a conserved quantity, that we call K; we see later that in quantum theory K should be an integer. One can find the effective theory for the coordinates ρ and z, by eliminating $\dot{\phi}$ by using the Routhian R [25], as follows:

$$\begin{aligned} L_{\rho z} &= -R \\ &= L - \dot{\phi} p_\phi \\ &= \frac{1}{4} m \left[1 + a f_1(r)\right] \dot{\rho}^2 + \frac{1}{4} m \dot{z}^2 \\ &- \frac{K^2}{m \rho^2 \left[1 + a \left(f_1(r) - \rho^2 f_2(r)\right)\right]}, \end{aligned} \tag{26}$$

in which we recognize the potential

$$V_{\text{eff}}(\rho, z) = \frac{K^2}{m \rho^2 \left[1 + a \left(f_1(r) - \rho^2 f_2(r)\right)\right]}. \tag{27}$$

It is useful to mention the properties of V_{eff}.
(1) It goes to $+\infty$ for $\rho = 0$ and $z \neq 0$.
(2) It is 0 on $\rho = z = 0$.
(3) It goes to $\pm\infty$ around the curve $g(\rho, z) := 1 + a\left(f_1(r) - \rho^2 f_2(r)\right) = 0$.

In Fig. 3 we have presented three plots of V_{eff} in the ρz-plane for $m = a = 1$, $m = 10a = 10$, and $a = 10m = 10$. We see that V_{eff} goes to $-\infty$ and $+\infty$ inside and outside the regions defined by the curve $g(\rho, z) = 0$, respectively.

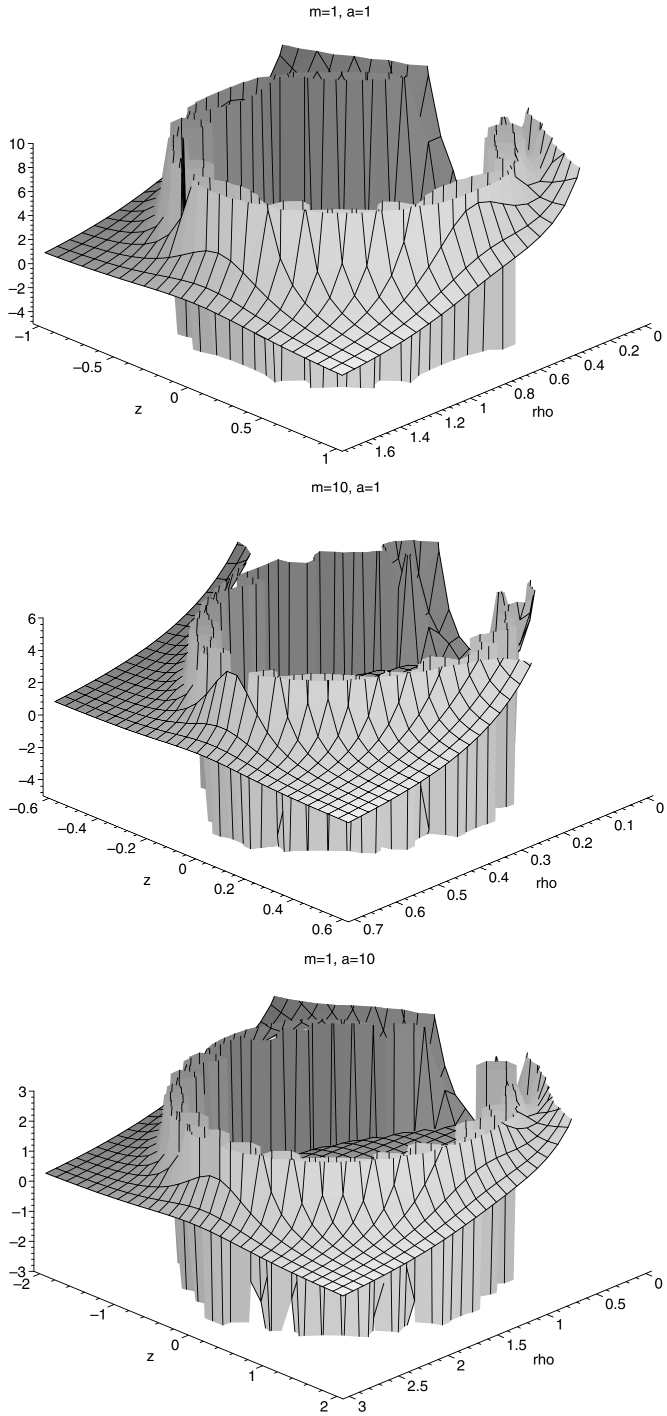

Fig. 1. Plots of V_{eff}, for $m = a = 1$, $m = 10a = 10$, and $a = 10m = 10$

We mention also, as the plots suggest, that the dynamics on the $z \equiv 0$-plane is unstable; that is, a small velocity $\dot{z} \neq 0$ hustles particles out of the $z = 0$-plane.

Before starting the discussion of quantum theory, let us have another look at the original Lagrangian (23). We mention that the Lagrangian is in the form of a pure kinetic term, represented by means of a metric $g_{ij}(x)$:

$$L = \frac{1}{2}\frac{m}{2} g_{ij}(x)\dot{x}^i\dot{x}^j, \tag{28}$$

in which $x^i = (\rho, \phi, z)$, and

$$\begin{aligned} g_{11}(\rho, z) &= 1 + a f_1(r), \\ g_{22}(\rho, z) &= \rho^2 \left[1 + a\left(f_1(r) - \rho^2 f_2(r)\right)\right], \\ g_{33} &= 1, g_{ij} = 0, i \neq j. \end{aligned} \tag{29}$$

We remind the reader that, although the Lagrangian is looking like a pure kinetic term, since one of the components of metric, g_{22}, changes sign, we can have negative energy states – among them there are bound states. By this interpretation of the Lagrangian, the Hamiltonian of the quantum theory is simply gained:

$$H_{2\mathrm{b}} = -\frac{1}{m}\nabla^2 = -\frac{1}{m}\frac{1}{\sqrt{|\det g|}}\partial_i\left[\sqrt{|\det g|}g^{ij}\partial_j\right], \tag{30}$$

in which $\det g$ is the determinant of g_{ij}. As g_{ij} is diagonal, $g^{ij} = 1/g_{ij}$, for $g^{ij} \neq 0$. We mention that the components of the metric are independent of the coordinate ϕ. Using the separation of variables, we choose the wave function $\Psi(\rho, z, \phi) = \psi(\rho, z)\Phi(\phi)$, with $\Phi(\phi) \propto \mathrm{e}^{\mathrm{i}l\phi}$, and due to the single-valuedness of the wave function, l should be an integer. We have finally

$$\begin{aligned} H^l_{2\mathrm{b}} = -\frac{1}{m}\Bigg\{ &\frac{1}{\sqrt{g_{11}|g_{22}|}}\partial_\rho\left[\sqrt{\frac{|g_{22}|}{g_{11}}}\partial_\rho\right] \\ &+ \frac{1}{\sqrt{g_{11}|g_{22}|}}\partial_z\left[\sqrt{g_{11}|g_{22}|}\partial_z\right] - \frac{l^2}{g_{22}}\Bigg\}, \end{aligned} \tag{31}$$

in which $H^l_{2\mathrm{b}}$ means the Hamiltonian for states with a specified value for l. By comparison, we see that the classical counterpart of the integer number l is K. Also we mention that $l^2/(mg_{22})$, as expected, is for V_{eff} in the quantum theory. Now let us choose a trial function $f(\rho, z)$ that vanishes outside the curve $g(\rho, z) = 0$. We consider the quantity

$$\begin{aligned} &\langle f|H^l_{2\mathrm{b}}|f\rangle \\ &= \int_{\text{inside } g(\rho,z)=0} f^*(\rho, z)\left(H^l_{2\mathrm{b}} f(\rho, z)\right)\sqrt{|\det g|}\mathrm{d}\rho\mathrm{d}z \\ &= A_{1,f} - l^2 A_{2,f} =: E_{f,l}, \end{aligned} \tag{32}$$

in which $A_{1,f}$ and $A_{2,f}$ are two numbers independent of l. We mention that, since $f(\rho, z)$ vanishes for $r \to \infty$, the contribution coming from the first two terms of $H^l_{2\mathrm{b}}$, taking into account the minus sign in front of it, is positive. The contribution from the last term of $H^l_{2\mathrm{b}}$, recalling the definition of $f(\rho, z)$, is negative. So for this kind of trial function, $A_{1,f}$ and $A_{2,f}$ are positive. Here we make a comment on the existence of bound states, at least for some ranges of l. We mention that for sufficiently large values of l, for a fixed trial function $f(\rho, z)$, $E_{f,l}$ can be negative. In fact one can, by increasing l, lower $E_{f,l}$ as much as wants. Now, by a variational theorem we know that $E_{f,l}$ is an upper limit for the lowest energy, and so we expect that for states with sufficient large l, there should be negative eigenvalues for Hamiltonian $H^l_{2\mathrm{b}}$. Denoting these negative eigenvalues by $E_{n,l}$, and the corresponding eigenfunctions by $\psi_{n,l}(\rho, z)$, with n for the possible quantum numbers, we have

$$\widetilde{\nabla}^2\psi_{n,l}(\rho, z) = \left(\frac{l^2}{g_{22}} - mE_{n,l}\right)\psi_{n,l}(\rho, z), \tag{33}$$

with $\widetilde{\nabla}^2$ as the Laplacian in the ρz-plane, given by

$$\begin{aligned} \widetilde{\nabla}^2 = &\frac{1}{\sqrt{g_{11}|g_{22}|}}\partial_\rho\left[\sqrt{\frac{|g_{22}|}{g_{11}}}\partial_\rho\right] \\ &+ \frac{1}{\sqrt{g_{11}|g_{22}|}}\partial_z\left[\sqrt{g_{11}|g_{22}|}\partial_z\right]. \end{aligned} \tag{34}$$

Now, since outside the curve $g(\rho, z) = 0$ the potential V_{eff} is positive definite, the coefficient of $\psi_{n,l}$ in the right-hand side is also positive. As $r \to \infty$ belongs to the outside of the curve $g(\rho, z) = 0$, by the properties of the spectrum of $\widetilde{\nabla}^2$, we expect $\psi_{n,l}|_{r\to\infty} \to 0$, that is, $\psi_{n,l}$ is representing a bound state. Physically we expect that for the negative eigenvalues, the wave function should be localized along the well inside the curve $g(\rho, z) = 0$, as g_{22} is approaching zero from below.

The manner we proved the existence of bound states can be used, by increasing l, for reasoning that there is no lowest energy state: the eigenvalues are unbounded from below. We recall that the potential (A.20) is obtained under the assumption that $\lambda \ll r$. As $\boldsymbol{\lambda} = \frac{1}{2}\mathbf{p} \times \boldsymbol{\theta}$, we see that for large values of the momentum, λ may be comparable, and even larger than r. One situation that might invalidate the assumption $\lambda \ll r$ can happen for very large values of l, corresponding to a large value of K in classical theory. In such cases one should consider the original potential (A.17). We recall that, although the absolute least energy is meaningless, to be found under the approximation $\lambda \ll r$, the least value of energy is still meaningful for states with a specified value for l.

The other issue concerns states with eigenvalues bigger than the maximum of the potential inside the curve $g(\rho, z) = 0$. We mention that an infinite tall wall has surrounded the inside region, and the question is if the wall can make for the possibility of forming bound states. In fact since the thickness of the wall behaves like $1/h$, with h as the height, by considerations coming from the WKB approximation for the tunnelling effect, we expect that the particles with positive energies can escape from the inside region. This situation is similar to the situation in the one-dimensional problem with potential $V(x) = 1/(x - x_0)$, for which by the WKB method one finds a n expression with a finite probability for tunnelling of positive energy particles.

As the final point, we make a comment on the possible values of the spin and l. The state of a two-photon system should be symmetric under the exchange of photons. A two-photon system can have 0, 1 and 2 as total spins, as for the first and last ones the spin states are symmetric, and for the second it is antisymmetric. Here the exchange of two photons means $z \to -z$ and $\phi \to \phi+\pi$. By considering the spatial dependence of the wave function, we have the following for the allowed spins and l:

$$\begin{aligned} S = 0, 2, \quad l = 0, 2, 4, \cdots \\ S = 1, \quad l = 1, 3, 5, \cdots \end{aligned} \tag{35}$$

4 Conclusion and discussion

We mention that the transformations of the gauge field as well as the field strength in a non-commutative space look like those of non-Abelian gauge theories. Besides we see that the action of non-commutative U(1) theory contains terms which are responsible for the interaction between photons, again as in the situation that we have in non-Abelian gauge theories. There is another observation that promotes the formal similarities of non-commutative and non-Abelian theories as to their behaviors, that is, the negative sign of the β-function, which makes manifest that these theories are asymptotically free [13,14]. The above mentioned observations make it reasonable to study whether and how the photons of non-commutative U(1) theory can make bound states. Also these observations make it reasonable to see if the techniques developed for QCD purposes can also be used for non-commutative U(1) theory. Here we used the so-called potential model, developed on the constituent gluon picture of QCD glue-balls. The basic ingredient of the potential model is that the self-interacting massless gauge particles may get a mass by the inclusion of non-perturbative effects. By calculating the amplitude for the scattering process between two massive photons, we extract the effective potential that is expected to capture the dynamics of the constituent photons. Using this effective potential, we formulate the Hamiltonian dynamics, by which arguments are presented in favor of the existence of photon bound states.

As possible photo-balls, like their glue-ball cousins, are non-perturbative in nature, it is expected that the lattice version of the non-commutative U(1) theory should appear as one of the natural ways to study a photo-ball's properties. It is remarkable that ordinary U(1) theory on the lattice develops an area law, suggesting a stringy picture for the force, for two charged particles [27]. There are suggestions for the lattice version of non-commutative gauge theories [28]. Specially, the finite N version of the theory is promising for numerical and simulation purposes. Recently, there have been a few works reporting the preliminaries results by the lattice version of the theories [29]. There are other suggestions for the non-perturbative definition of the non-commutative U(1) gauge theory [30].

By the current experiments there has not been any signal for possible non-commutativity. So the common expectation is that the evidence for non-commutativity, if any, should modify the processes that occur at energies much higher than those presently available. This is why by present experimental data one can just suggest an upper limit for non-commutative effects. There has been another suggestion that non-commutativity effects may appear due to applying of a sufficiently strong magnetic field on samples containing moving charged particles. It would be extremely interesting if the non-commutative view would let us learn something new about relevant phenomena [31].

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Appendix A: Massive non-commutative U(1) theory and effective potential between photons

A.1 Massive photon–photon scattering amplitude

Here, following the procedure developed for the QCD case, we give mass to the photons of non-commutative U(1) theory. As described this is done by introducing an extra scalar field, getting the Lagrangian density

$$L = L_{\text{gauge-field}} + \frac{m^2}{2}\left(A_\mu + \frac{\mathrm{i}}{e}\partial_\mu V(\varphi) \star V^{-1}(\varphi)\right)^2_\star \tag{A.1}$$

in which $(\cdots)^2_\star = (\cdots)\star(\cdots)$, and $V(\varphi)$ is the $\star$-phase defined by the scalar field φ; see (16). The action defined by the above Lagrangian is invariant under transformations:

$$\begin{aligned} A'_\mu(x) &= U \star A_\mu(x) \star U^{-1} + \frac{\mathrm{i}}{e} U \star \partial_\mu U^{-1}, \\ V(\varphi') &= U \star V(\varphi), \end{aligned} \tag{A.2}$$

in which U is the same in (15). Now we just list the Feynman rules [2,14,18]. For the propagator one easily takes the one for a massive vector field:

$$\mathrm{i}D^{\mu\nu}(p) = \frac{-\mathrm{i}g^{\mu\nu}}{p^2 - m^2} + \frac{\mathrm{i}p^\mu p^\nu}{(p^2 - m^2)m^2}. \tag{A.3}$$

In the non-relativistic limit we have for the momentum and polarization vectors [4,5]

$$p^\mu = \left(m + \frac{\mathbf{p^2}}{2m}, \mathbf{p}\right), \epsilon^\mu = \left(\frac{\mathbf{p}\cdot\mathbf{e}}{m}, \mathbf{e} + \frac{\mathbf{p}\cdot\mathbf{e}}{2m^2}\mathbf{p}\right), \tag{A.4}$$

in which $\mathbf{e}$ is a 3-vector satisfying $\mathbf{e}^* \cdot \mathbf{e} = 1$. From the Lorentz condition [3–5], we have $p \cdot \epsilon = p^\mu \epsilon_\mu = 0$. In this work we assume for the signature of the metric $g_{\mu\nu} = (+1, -1, -1, -1)$. As in this work we restrict ourselves to tree diagrams, after removing one $\star$, and by the Lorentz condition, we practically are using the Lagrangian [2]

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu, \tag{A.5}$$

with $F_{\mu\nu}$ defined in (14). There are three- and four-photon vertices given in Fig. 1. As in this work we consider non-commutativity just in spatial directions, that is assuming

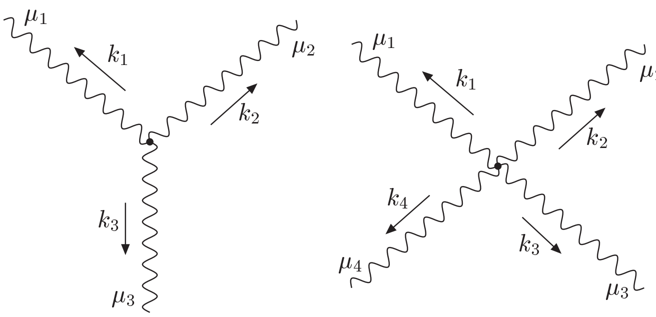

Fig. 2. Three- and four-photon vertices of non-commutative U(1) theory

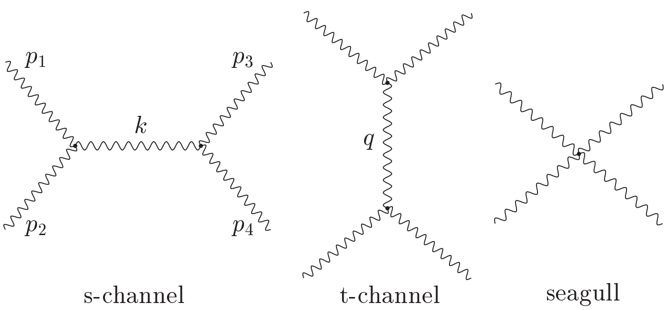

Fig. 3. s-channel, t-channel, and seagull diagrams

$\theta^{0i} = 0$ for $i = 1, 2, 3$, we have for the vertex functions [14, 18]

$$\begin{aligned} \Gamma^{\mu_1\mu_2\mu_3}_{k_1,k_2,k_3} &= -2e \sin\left(\frac{\mathbf{k}_1 \ltimes \mathbf{k}_2}{2}\right) \\ &\times [(k_1 - k_2)^{\mu_3} g^{\mu_1\mu_2} + (k_2 - k_3)^{\mu_1} g^{\mu_2\mu_3} \\ &\quad +(k_3 - k_1)^{\mu_2} g^{\mu_3\mu_1}] \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} &\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{k_1,k_2,k_3,k_4} \\ &= -4\mathrm{i}e^2 \left[\sin\left(\frac{\mathbf{k}_1 \ltimes \mathbf{k}_2}{2}\right) \sin\left(\frac{\mathbf{k}_3 \ltimes \mathbf{k}_4}{2}\right)\right. \\ &\qquad \times (g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}) \\ &+ \sin\left(\frac{\mathbf{k}_3 \ltimes \mathbf{k}_1}{2}\right) \sin\left(\frac{\mathbf{k}_2 \ltimes \mathbf{k}_4}{2}\right) \\ &\qquad \times (g^{\mu_1\mu_4} g^{\mu_2\mu_3} - g^{\mu_1\mu_2} g^{\mu_3\mu_4}) \\ &+ \sin\left(\frac{\mathbf{k}_1 \ltimes \mathbf{k}_4}{2}\right) \sin\left(\frac{\mathbf{k}_2 \ltimes \mathbf{k}_3}{2}\right) \\ &\qquad \left.\times (g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_3} g^{\mu_2\mu_4})\right], \end{aligned} \tag{A.7}$$

in which $\mathbf{a} \ltimes \mathbf{b} \equiv \theta^{ij} a_i b_j$, and the momenta and indices are given in Fig. 1. Also in each vertex energy-momentum conservation should be understood.

Although there are four diagrams at tree level, those coming from the s-, t- and u-channels, and the seagull diagram (Fig. 2), when extracting the potential, by the properly symmetrized wave function for identical particle systems, the "exchange" or "symmetry" diagrams are automatically taken care of, causing that only one of the t- and u-channels' contributions should be added to the others' contributions [4].

Comparing QCD vertex functions with those of noncommutative theory shows us that the only difference between two theories is in pre-factors. This is the cause that those parts of the calculations concerning the kinematical parts are similar to those done for glue-balls [3–5], and consequently here we concentrate on differences between two theories and results. First, exactly as in the QCD case, the leading order contribution of the s-channel is negligible in the non-relativistic limit.

Now we come to the t-channel,

$$\begin{aligned} &\mathrm{i}\mathcal{M}^t_{fi} \\ &= -4e^2 \sin\left(\frac{\mathbf{p_1} \ltimes \mathbf{q}}{2}\right) \sin\left(\frac{\mathbf{p_2} \ltimes \mathbf{q}}{2}\right) \\ &\times [g_{\mu\lambda}(p_1 + p_3)_\rho + g_{\lambda\rho}(p_1 - 2p_3)_\mu \\ &\qquad + g_{\rho\mu}(p_3 - 2p_1)_\lambda] \\ &\times \epsilon^\mu_1 \epsilon^{*\lambda}_3 \frac{-\mathrm{i}\left(g^{\rho\delta} - \frac{q^\rho q^\delta}{m^2}\right)}{q^2 - m^2} \epsilon^\nu_2 \epsilon^{*\sigma}_4 \\ &\times [g_{\nu\sigma}(p_2 + p_4)_\delta + g_{\sigma\delta}(p_2 - 2p_4)_\nu \\ &\qquad + g_{\delta\nu}(p_4 - 2p_2)_\sigma], \end{aligned} \tag{A.8}$$

in which $q = p_3 - p_1 = p_2 - p_4$. We continue in the center-of-mass frame, defined by

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}_i, \mathbf{p}_3 = -\mathbf{p}_4 = \mathbf{p}_f, \mathbf{q} = \mathbf{p}_f - \mathbf{p}_i. \tag{A.9}$$

The result in the non-relativistic limit is [3–5]

$$\begin{aligned} \mathrm{i}\mathcal{M}^t_{fi} &= 4\mathrm{i}e^2 \frac{\sin^2\left(\frac{\mathbf{p} \ltimes \mathbf{q}}{2}\right)}{\mathbf{q}^2 + m^2} \\ &\times \left[4m^2 + 3\mathbf{q}^2 - 2\mathbf{S}^2\mathbf{q}^2 + 2(\mathbf{S}\cdot\mathbf{q})^2 + 6\mathrm{i}\mathbf{S}\cdot(\mathbf{q}\times\mathbf{p})\right] \\ &+ O(\mathbf{p}^2), \end{aligned} \tag{A.10}$$

in which $\mathbf{S}$ is the total spin operator for the two-photon system. We mention that the kinematical dependence of the t-channel amplitude, given by the terms $[\cdots]$, not surprisingly is exactly that for gluons, presented in (7). In fact the only difference between the case of gluons and photons in non-commutative U(1) theory, as mentioned above, is in the pre-factor, originating from the difference between the structure constants of the group that appear in the vertex functions [14].

Now we come to the seagull diagram, with the contribution

$$\begin{aligned} \mathrm{i}\mathcal{M}^{\text{s.g.}}_{fi} &= -4\mathrm{i}e^2 \epsilon^\mu_1 \epsilon^\nu_2 \epsilon^{*\lambda}_3 \epsilon^{*\sigma}_4 \qquad\qquad \text{(A.11)} \\ &\times \left[\sin\left(\frac{\mathbf{p_1} \ltimes \mathbf{p_2}}{2}\right) \sin\left(\frac{\mathbf{p_3} \ltimes \mathbf{p_4}}{2}\right) (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda})\right. \\ &+ \sin\left(\frac{\mathbf{p_3} \ltimes \mathbf{p_1}}{2}\right) \sin\left(\frac{\mathbf{p_2} \ltimes \mathbf{p_4}}{2}\right) (g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\nu\lambda}) \\ &+ \left.\sin\left(\frac{\mathbf{p_1} \ltimes \mathbf{p_4}}{2}\right) \sin\left(\frac{\mathbf{p_2} \ltimes \mathbf{p_3}}{2}\right) (g^{\mu\nu} g^{\mu\lambda} - g^{\mu\lambda} g^{\nu\sigma})\right]. \end{aligned}$$

By doing the manipulations we get

$$\mathrm{i}\mathcal{M}_{fi}^{\mathrm{s.g.}}=8\mathrm{i}e^2\sin\left(\frac{\mathbf{p}_1\ltimes\mathbf{p}_3}{2}\right)\sin\left(\frac{\mathbf{p}_2\ltimes\mathbf{p}_4}{2}\right)\left(\frac{1}{2}\mathbf{S}^2-2\right). \tag{A.12}$$

Now we mention that the contribution of the seagull channel, for a small non-commutativity parameter, is something proportional to $(\theta\mathbf{p})^2\,\mathbf{p^2}$ which is of the order of $\mathbf{p}^2$, which we ignore. This observation is different from that for QCD glue-balls: for them the contribution of the seagull diagram is in zeroth order of momentum, and thus should be kept. The seagull's contribution appears to be in the form of $\delta(\mathbf{r})$ in the potential.

A.2 Effective potential between photons

Before we proceed, we define the vector $\boldsymbol{\theta}$ based on the tensor θ^{ij} by $\theta^i\equiv\epsilon^{ijk}\theta_{jk}$. By this vector we can write the $\ltimes$-product as

$$\begin{aligned}\mathbf{a}\ltimes\mathbf{b}&=\theta^{ij}a^ib^j=a^i\frac{1}{2}\epsilon_{lij}\theta^lb^j\\&=\frac{1}{2}\boldsymbol{\theta}\cdot(\mathbf{a}\times\mathbf{b})=\frac{1}{2}\mathbf{b}\cdot(\boldsymbol{\theta}\times\mathbf{a}).\end{aligned} \tag{A.13}$$

By this we have for the t-channel contribution

$$\mathrm{i}\mathcal{M}_{fi}^t=4\mathrm{i}e^2\frac{\sin^2\left(\frac{1}{2}\mathbf{q}\cdot\boldsymbol{\lambda}\right)}{\mathbf{q}^2+m^2}\Upsilon(\mathbf{q}), \tag{A.14}$$

in which $\boldsymbol{\lambda}=\frac{1}{2}\mathbf{p}\times\boldsymbol{\theta}$ and $\Upsilon(\mathbf{q})=4m^2+3\mathbf{q}^2-2\mathbf{S}^2\mathbf{q}^2+2(\mathbf{S}\cdot\mathbf{q})^2+6\mathrm{i}\mathbf{S}\cdot(\mathbf{q}\times\mathbf{p})$. By the total amplitude the potential can be deduced using (6)

$$\begin{aligned}V_{2\gamma}(\mathbf{r})=&-\frac{e^2}{m^2}\int\frac{\mathrm{d}^3q}{8\pi^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2+m^2}\\&\times\sin^2\left(\frac{1}{2}\mathbf{q}\cdot\boldsymbol{\lambda}\right)\Upsilon(\mathbf{q}).\end{aligned} \tag{A.15}$$

By writing the exponential form of sin(...), and defining

$$U(R):=\int\frac{\mathrm{d}^3q}{8\pi^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2+m^2}=\frac{\mathrm{e}^{-mR}}{4\pi R}, \tag{A.16}$$

with $R=|\mathbf{R}|$, and by $\mathbf{q}\to-\mathrm{i}\boldsymbol{\nabla}$, we have

$$V_{2\gamma}(\mathbf{r})=-\frac{e^2}{4m^2}\Upsilon(-\mathrm{i}\boldsymbol{\nabla})\left[2U(r)-U(r_+)-U(r_-)\right], \tag{A.17}$$

with $\mathbf{r}_\pm=\mathbf{r}\pm\boldsymbol{\lambda}$. We mention that, for $\boldsymbol{\lambda}=\mathbf{0}$, the potential vanishes; this happens in the following cases:
(1) $\boldsymbol{\theta}=\mathbf{0}$,
(2) $\mathbf{p}=\mathbf{0}$, and
(3) $\mathbf{p}\parallel\boldsymbol{\theta}$. It is reasonable to see this behavior of the potential for small non-commutativity parameter, defined here by $\lambda\ll r$ and $\lambda m\ll 1$. In this limit, the first surviving terms are given by

$$V_{2\gamma}(\mathbf{r})=\frac{e^2}{4m^2}\Upsilon(-\mathrm{i}\boldsymbol{\nabla})(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})^2U(r)+O(\lambda^4). \tag{A.18}$$

Recalling that for a function $f(r)$, $\partial_i f(r)=x_i\nabla_r f$, with $\nabla_r=r^{-1}\partial_r$, and using

$$\begin{aligned}(\mathbf{p}\times\mathbf{S})\cdot\mathbf{r}=(\mathbf{r}\times\mathbf{p})\cdot\mathbf{S}=\mathbf{L}\cdot\mathbf{S},\\ \nabla^2U(r)=m^2U(r)-\delta(\mathbf{r}),\end{aligned} \tag{A.19}$$

with $\mathbf{L}$ the total angular momentum, we get the expression for the potential

$$\begin{aligned}V_{2\gamma}(\mathbf{r})=&\frac{e^2}{4m^2}\Bigg\{m^2\left(1+2S^2\right)\left[\lambda^2\nabla_r+(\boldsymbol{\lambda}\cdot\mathbf{r})^2\nabla_r\nabla_r\right]\\&-2\Bigg[\left[S^2\lambda^2+2(\boldsymbol{\lambda}\cdot\mathbf{S})^2\right]\nabla_r\nabla_r\\&+(\boldsymbol{\lambda}\cdot\mathbf{r})^2(\mathbf{S}\cdot\mathbf{r})^2\nabla_r\nabla_r\nabla_r\nabla_r\\&+\left[4(\boldsymbol{\lambda}\cdot\mathbf{S})(\boldsymbol{\lambda}\cdot\mathbf{r})(\mathbf{S}\cdot\mathbf{r})+\lambda^2(\mathbf{S}\cdot\mathbf{r})^2+S^2(\boldsymbol{\lambda}\cdot\mathbf{r})^2\right]\\&\qquad\times\nabla_r\nabla_r\nabla_r\Bigg]\\&+6\Big[\left[\lambda^2\nabla_r\nabla_r+(\boldsymbol{\lambda}\cdot\mathbf{r})^2\nabla_r\nabla_r\nabla_r\right](\mathbf{L}\cdot\mathbf{S})\\&\qquad+2(\mathbf{p}\times\mathbf{S})\cdot\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\mathbf{r})\nabla_r\nabla_r\Big]\Bigg\}U(r)\\&+\mathrm{D.D.}+O(\lambda^4),\end{aligned} \tag{A.20}$$

in which $\lambda=|\boldsymbol{\lambda}|$, $S=|\mathbf{S}|$, and D.D. is for the distributional derivatives of the function $U(r)$, containing a δ-function and its derivatives; we calculate and present the explicit expression of D.D. in Appendix B.

We now make some comments on the potential given by (A.20). First, we mention that due to the $\mathbf{r}$'s in the inner products, the effective lowest power is r^{-5}. Second, the strength of the potential, through the definition of $\boldsymbol{\lambda}$, depends on the momentum. Third, let us consider the spin-independent part of the potential, that is, setting $S=0$,

$$\begin{aligned}V_{2\gamma}^{S=0}(\mathbf{r})=&\frac{e^2}{4}\frac{\mathrm{e}^{-mr}}{4\pi}\\&\times\left[-\lambda^2\frac{mr+1}{r^3}+(\boldsymbol{\lambda}\cdot\hat{\mathbf{r}})^2\frac{m^2r^2+3mr+3}{r^3}\right].\end{aligned} \tag{A.21}$$

We mention that the $m=0$ limit of the above expression is well defined. It is known that in non-commutative field theories particles behave as electric dipoles [16,18,22–24]. The electric dipole depends on the strength of the non-commutativity parameter as well as the momentum, and is perpendicular to both of them; it is given by $\mathbf{d}=\frac{1}{4}e\boldsymbol{\theta}\times\mathbf{p}$. For the two-photon system, in the center-of-mass frame, for which $\mathbf{p}_1=-\mathbf{p}_2=\mathbf{p}$, we have $\mathbf{d}_1=-\mathbf{d}_2=\mathbf{d}$. The potential for a system of two electric dipoles $\mathbf{d}_1$ and $\mathbf{d}_2$ is given by

$$V_{\mathrm{dipoles}}(\mathbf{r})=\frac{1}{4\pi}\frac{1}{r^3}\left[\mathbf{d}_1\cdot\mathbf{d}_2-3(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\right]. \tag{A.22}$$

We see that the two expressions (A.21) and (A.22) are equivalent for $m=0$ and $\mathbf{d}=\frac{1}{2}e\boldsymbol{\lambda}$. In fact the expression (A.21) is the potential of two anti-parallel dipoles in a theory in which the potential of a charged particle is given by the so-called Yukawa potential: $V(r)=\frac{e}{4\pi}\mathrm{e}^{-mr}/r$. In principle, one could justify that the potential (A.20) is in fact that for two anti-parallel dipoles, included by spin–orbit and spin–dipole interactions in a Yukawa-type theory. Finally, we mention

$$\boldsymbol{\lambda}\cdot\mathbf{r}=\frac{1}{2}(\mathbf{p}\times\boldsymbol{\theta})\cdot\mathbf{r}=\frac{1}{2}\boldsymbol{\theta}\cdot(\mathbf{r}\times\mathbf{p})=\frac{1}{2}\boldsymbol{\theta}\cdot\mathbf{L}, \quad \text{(A.23)}$$

that can be inserted in the relevant parts of the potential (A.20). It is the famous θ–L coupling, previously found in studies concerning the implications of non-commutativity in low energy phenomena [16,18,23].

Appendix B: Distributional derivatives

Here we calculate the distributional derivatives [26]. First we consider $\partial_i\partial_j\frac{\mathrm{e}^{-mr}}{r}$. The distributional derivative can be calculated by its effect on a test function $\phi(\mathbf{r})$:

$$\begin{aligned}\langle\partial_i\partial_j\frac{\mathrm{e}^{-mr}}{r},\phi\rangle &:= \langle\frac{\mathrm{e}^{-mr}}{r},\partial_i\partial_j\phi\rangle=\int\frac{\mathrm{e}^{-mr}}{r}\partial_i\partial_j\phi(\mathbf{r})\mathrm{d}^3r\\ &=\lim_{\varepsilon\to0^+}\int_{r\geq\varepsilon}\frac{\mathrm{e}^{-mr}}{r}\partial_i\partial_j\phi(\mathbf{r})\mathrm{d}^3r, \end{aligned}\quad \text{(B.1)}$$

in which $\mathrm{d}^3r=r^2\mathrm{d}r\mathrm{d}\Omega$. The limit above does exist because the integral in the second line, due to r^2 in d^3r, is finite. $r=0$ is excluded from the last integral, and so we can do integrations by parts:

$$\begin{aligned}&I_{ij}\\ &=\lim_{\varepsilon\to0^+}\left[\int_{r\geq\varepsilon}\partial_i\left(\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi\right)\mathrm{d}^3r-\int_{r\geq\varepsilon}\partial_i\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi\mathrm{d}^3r\right]\\ &=\lim_{\varepsilon\to0^+}\left[\int_{r\geq\varepsilon}\boldsymbol{\nabla}\cdot\left(\hat{\mathbf{e}}_i\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi\right)\mathrm{d}^3r-\int_{r\geq\varepsilon}\partial_i\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi\mathrm{d}^3r\right]\\ &=\lim_{\varepsilon\to0^+}\left[\int_{r=\varepsilon}\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi(-\hat{\mathbf{e}}_i\cdot\hat{\mathbf{r}})r^2\mathrm{d}\Omega\right. \qquad \text{(B.2)}\\ &\qquad\left.-\int_{r\geq\varepsilon}\partial_i\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi\mathrm{d}^3r\right]\\ &=\lim_{\varepsilon\to0^+}\left[-\int_{r=\varepsilon}\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi n_ir^2\mathrm{d}\Omega-\int_{r\geq\varepsilon}\partial_i\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi\mathrm{d}^3r\right],\end{aligned}$$

in which $\hat{\mathbf{e}}_i$ is for a unit vector in the x, y and z directions, and $\hat{\mathbf{r}}=(n_1,n_2,n_3)$. The first integral in last line is proportional to ε and so vanishes in the limit. So we get

$$I_{ij}=-\lim_{\varepsilon\to0^+}\int_{r\geq\varepsilon}\partial_i\frac{\mathrm{e}^{-mr}}{r}\partial_j\phi\mathrm{d}^3r. \quad \text{(B.3)}$$

By repeating the steps above, we arrive at

$$I_{ij}=-\frac{4\pi}{3}\phi(0)\delta_{ij}+\lim_{\varepsilon\to0^+}\int_{r\geq\varepsilon}\phi\partial_i\partial_j\frac{\mathrm{e}^{-mr}}{r}\mathrm{d}^3r; \quad \text{(B.4)}$$

for getting it we used $\partial_if(r)=n_i\partial_rf(r)$, and also by keeping the integrands to the first non-vanishing order in ε, and using

$$\int n_in_j\mathrm{d}\Omega=\frac{4\pi}{3}\delta_{ij}. \quad \text{(B.5)}$$

The limit above exists, by using the fact that the value of the function at the origin is constant and independent from $\Omega=\Omega(\theta,\varphi)$, and recalling

$$\int(3n_in_j-\delta_{ij})\mathrm{d}\Omega=0. \quad \text{(B.6)}$$

One can remove the limit by respecting the order of integrations. By all this we get

$$\partial_i\partial_j\frac{\mathrm{e}^{-mr}}{r}\to-\frac{4\pi}{3}\delta_{ij}\delta(\mathbf{r})+\mathrm{pf}\left[\partial_i\partial_j\frac{\mathrm{e}^{-mr}}{r}\right], \quad \text{(B.7)}$$

in which "pf" stands for a pseudo-function, which here simply means that in integrals the integration on a solid angle should be done before a radial one.

By repeating the procedure above we obtain

$$\begin{aligned}&\partial_i\partial_j\partial_k\frac{\mathrm{e}^{-mr}}{r}\to \qquad \text{(B.8)}\\ &-\frac{4\pi}{5}\left[\delta_{ij}\partial_k+\delta_{jk}\partial_i+\delta_{ki}\partial_j\right]\delta(\mathbf{r})+\mathrm{pf}\left[\partial_i\partial_j\partial_k\frac{\mathrm{e}^{-mr}}{r}\right]\end{aligned}$$

and

$$\begin{aligned}&\partial_i\partial_j\partial_k\partial_l\frac{\mathrm{e}^{-mr}}{r}\to\\ &-\frac{4\pi}{15}\left[\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right]\delta(\mathbf{r})\\ &\qquad-\frac{10\pi}{51}\left[\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right]\nabla^2\delta(\mathbf{r})\\ &-\frac{20\pi}{51}\left[\delta_{ij}\partial_k\partial_l+\delta_{ik}\partial_j\partial_l+\delta_{il}\partial_j\partial_k+\delta_{kl}\partial_i\partial_j\right.\\ &\qquad\left.+\delta_{jl}\partial_i\partial_k+\delta_{kj}\partial_i\partial_l\right]\delta(\mathbf{r})\\ &+\mathrm{pf}\left[\partial_i\partial_j\partial_k\partial_l\frac{\mathrm{e}^{-mr}}{r}\right], \qquad \text{(B.9)}\end{aligned}$$

in which again "pf" simply means that in integrals the integration on a solid angle should be done before a radial one. The combination $\nabla^2\partial_i\partial_j$ is simply $\delta^{kl}\partial_i\partial_j\partial_k\partial_l$.

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